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The stress field in a matrix containing a partially debonded elliptic inhomogeneity of identical Poisson's ratio

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Studies on the stress field in an infinite elastic matrix containing an elliptic inhomogeneity that has debonded over an arc of its boundary are reported. The matrix is under plane strain conditions. The solution is obtained by an extension of Eshelby's equivalent inclusion technique, but is restricted to the case when Poisson's ratios of the matrix and inhomogeneity are equal. Numerical results are given for the stress-intensity factors at the tips of the debonded arc and for the relative displacements across the debond.

1. Introduction

The mechanical properties of composites, especially their fracture toughness, are strongly influenced by the quality of bond between the matrix and reinforcing inhomogeneities. Yet, given the fact that the latter generally act as stress raisers, debonding seems to be inevitable. The study of the behaviour of a composite in which the inhomogeneities have debonded partially or fully from the matrix is therefore important in gaining an understanding of its fracture toughness.

The mathematical problem corresponding to partially or fully debonded inhomogeneities in an elastic matrix is far more complex to analyse than the already tedious problem resulting from perfectly bonded inhomogeneities [1–5]. In a recent investigation [6] (this paper also contains other useful references), the present authors calculated the stress field in an infinite, elastic matrix containing a partially debonded elliptic inhomogeneity. The matrix was under antiplane strain conditions. The solution was obtained by an extension of Eshelby's equivalent inclusion technique [7, 8]. In this paper the technique is further extended to the study of a partially debonded inhomogeneity under plane strain conditions. The extension is by no means straightforward, although several simplifications result from equality of Poisson's ratios between the matrix and inhomogeneity. It is this special case which forms the subject of the present paper. The general case of arbitrary Poisson's ratios requires an altogether different approach; this solution will be reported in a separate communication.

From the solution of the stress field we calculate the stress intensity factors at the tips of the debonded arc and the relative displacements across the debond. Two numerical examples have been studied in detail.

2. Statement of the problem

Fig. 1 shows the geometry of the matrix along with the elliptic inhomogeneity (Ω) and the co-ordinate axes. (λ, μ) and (λ_1, μ_1) denote the Lamé constants of the matrix and inhomogeneity, respectively. The solution of the equation of equilibrium

$$\sigma_{ij,j}(\boldsymbol{u}) + X_i = 0; \qquad i, j = 1, 2 \qquad (1)$$

can be formally represented as

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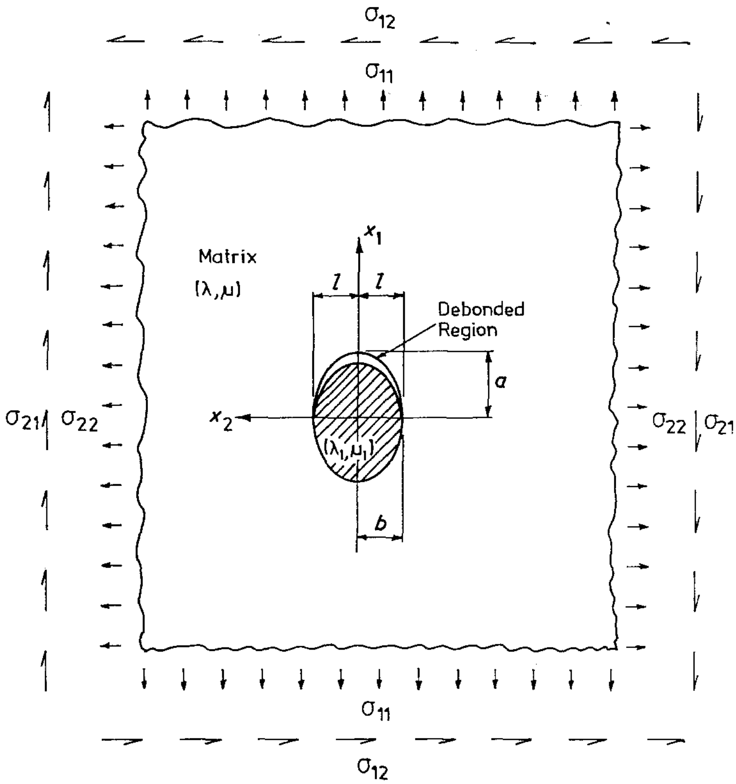

Figure 1 The debonded elliptic inhomogeneity, showing the coordinate axes and the remote stress-field.

$$u_{\mathrm{m}}(\boldsymbol{x}) = -\int_{\Omega}\sigma^{*}_{ij,j}(\boldsymbol{x}')G_{im}(\boldsymbol{x}, \boldsymbol{x}')\mathrm{d}\Omega(\boldsymbol{x}')$$

$$-\fint_{\Gamma}\gamma_i(\boldsymbol{x}')\sigma_{ij}[\boldsymbol{G}^m(\boldsymbol{x}, \boldsymbol{x}')]n_j(\boldsymbol{x}')\mathrm{d}\Gamma(\boldsymbol{x}') \quad (2)$$

Here, Γ denotes the arc of the debonded boundary, σ^*_{ij} are the eigenstresses associated with the (unknown) eigenstrains e^*_{ij} distributed over the region of $\boldsymbol{\Omega}$, and $\gamma_i(\boldsymbol{x})$ are the dislocation densities across the debond for a homogeneous material. The unknowns σ^*_{ij} and $\gamma_i(\boldsymbol{x})$ are to be determined from Eshelby's equivalence relation

$$\sigma^*_{ij}(\boldsymbol{x}) = \bar{\sigma}_{ij}(\boldsymbol{u}) + \bar{\sigma}^0_{ij}(\boldsymbol{x}), \qquad (\boldsymbol{x} \subset \boldsymbol{\Omega}) \quad (3)$$

and the traction free boundary condition across the debond

$$\{\sigma_{ij}(\boldsymbol{u}) + \sigma^0_{ij}\}n_j(\boldsymbol{x}) = 0, \qquad (\boldsymbol{x} \subset \Gamma) \quad (4)$$

where σ^0_{ij} is the far-field applied stress, $n_j(\boldsymbol{x})$ is the unit outward normal to Γ at $\boldsymbol{x}$, and $\bar{\sigma}_{ij}$ is the stress-operator with (λ, μ) replaced by $(\bar{\lambda}, \bar{\mu})$ where

$$\bar{\lambda} = (\lambda - \lambda_1), \qquad \bar{\mu} = (\mu - \mu_1) \quad (5)$$

Finally, Green's function $G_{ij}(\boldsymbol{x}, \boldsymbol{x}')$ in Equation 2 is given by

$$G_{ij}(\boldsymbol{x}, \boldsymbol{x}') = \left[\frac{\bar{x}_i\bar{x}_j}{\bar{R}^2} - (3 - 4\nu)\delta_{ij}\log\bar{R}\right]/8\pi\mu(1 - \nu) \quad (6)$$

where

$$\bar{x}_i = (x'_i - x_i) \quad (7)$$

$$\bar{R} = |\boldsymbol{x} - \boldsymbol{x}'| \quad (8)$$

and ν is Poisson's ratio.

3. Solution of Equations 3 and 4

Due to debonding, the eigenstrains e^*_{ij} will consist of both singular and non-singular parts, the former being absent for perfect bonding. Let us assume that

$$e^*_{ij} = e^{*\mathrm{I}}_{ij} + e^{*\mathrm{II}}_{ij}$$

and correspondingly

$$\sigma^*_{ij} = \sigma^{*\mathrm{I}}_{ij} + \sigma^{*\mathrm{II}}_{ij}$$

where

$$\sigma^*_{ij} = \lambda\,\delta_{ij}(e^*_{11} + e^*_{22}) + 2\mu\,e^*_{ij} \quad (9)$$

Here, $e^{*\mathrm{I}}_{ij}$ and $\sigma^{*\mathrm{I}}_{ij}$ denote the non-singular parts which will be in the form of a power-series like the applied strains

$$e_{ij}^{0} = \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} a_{\alpha\beta}^{ij} x_1^{\alpha} x_2^{\beta} \quad (10)$$

Therefore we assume that

$$e_{ij}^{*\mathrm{I}} = \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} b_{\alpha\beta}^{ij} x_1^{\alpha} x_2^{\beta} \quad (11)$$

The expansion coefficients $b_{\alpha\beta}^{ij}$ are to be found in terms $a_{\alpha\beta}^{ij}$ by equating the non-singular parts of Equation 3. The corresponding equation is

$$\sigma_{ij}^{*\mathrm{I}}(\boldsymbol{x}) = \bar{\sigma}_{ij}^{(0)}(\boldsymbol{x}) + \int_{\Omega} \sigma_{kk'}^{*\mathrm{I}}(x')\bar{\sigma}_{ij}[\boldsymbol{G}^{k}(\boldsymbol{x}, \boldsymbol{x}')]_{,k'} \mathrm{d}\Omega(\boldsymbol{x}'); \quad (\boldsymbol{x} \subset \Omega) \quad (12)$$

where we have incorporated an integration by parts in the integral over Ω.

The integral over Ω in Equation 12 can be evaluated as in our earlier work [6] and we can write it as

$$\int_{\Omega} \sigma_{kk'}^{*\mathrm{I}}(\boldsymbol{x}')\bar{\sigma}_{ij}[\boldsymbol{G}^{k}(\boldsymbol{x}, \boldsymbol{x}')]_{,k'} \mathrm{d}\Omega(\boldsymbol{x}') = \sum_{\alpha,\beta} \bar{C}_{\alpha\beta}^{ij} x_1^{\alpha} x_2^{\beta} \quad (13)$$

where $\bar{C}_{\alpha\beta}^{ij}$ are linear expressions in terms of $b_{\alpha\beta}^{ij}$. The details of evaluating Equation 13 are given in the Appendix. The coefficients $\bar{C}_{\alpha\beta}^{ij}$ can be written in a more general form than that in the Appendix:

$$\bar{C}_{\alpha\beta}^{ij} = B_{pq}^{kk'} \bar{D}_{kk'ij}^{pq\alpha\beta} \quad (14)$$

where

$$B_{\alpha\beta}^{ij} = \lambda\,\delta_{ij}(b_{\alpha\beta}^{11} + b_{\alpha\beta}^{22}) + 2\mu b_{\alpha\beta}^{ij} \quad (15)$$

Summation convention over repeated indices is assumed throughout this work. Note that $B_{\alpha\beta}^{ij}$ are the expansion coefficients of $\sigma_{ij}^{*\mathrm{I}}$ corresponding to $e_{ij}^{*\mathrm{I}}$ (Equation 11).

A comparison of like terms of the expansions on both sides of Equation 13 leads to the following equations for $B_{\alpha\beta}^{ij}$:

$$B_{\alpha\beta}^{ij} = \bar{A}_{\alpha\beta}^{ij} + B_{pq}^{kk'} \bar{D}_{kk'ij}^{pq\alpha\beta} \quad (16)$$

where

$$A_{\alpha\beta}^{ij} = \lambda\,\delta_{ij}(a_{\alpha\beta}^{11} + a_{\alpha\beta}^{22}) + 2\mu\, a_{\alpha\beta}^{ij}$$
$$\bar{A}_{\alpha\beta}^{ij} = \bar{\lambda}\,\delta_{ij}(a_{\alpha\beta}^{11} + a_{\alpha\beta}^{22}) + 2\bar{\mu}\, a_{\alpha\beta}^{ij} \quad (17)$$

with $\bar{\lambda}$, $\bar{\mu}$ defined in Equation 5.

Thus $(B_{\alpha\beta}^{ij})$ can be solved from Equation 16 in terms of $(\bar{A}_{\alpha\beta}^{ij})$ which are known in terms of $(a_{\alpha\beta}^{ij})$ from Equation 17. Then Equation 15 gives $(b_{\alpha\beta}^{ij})$, thus completing the determination of the non-singular eigenstrains $e_{ij}^{*\mathrm{I}}$.

3.1. Singular part of the eigenstrains

Next consider the singular part of the eigenstrains $e_{ij}^{*\mathrm{II}}$. From Equation 3 it follows that these strains must satisfy the following equivalence relation

$$\sigma_{ij}^{*\mathrm{II}}(\boldsymbol{x}) = \bar{\sigma}_{ij}(\boldsymbol{I}) - \oint_{\Gamma} \gamma_k(\boldsymbol{x}')n_{k'}(\boldsymbol{x}')\sigma_{kk'}\{\bar{\sigma}_{ij}[\mathbf{G}(\boldsymbol{x}, \boldsymbol{x}')]\}\mathrm{d}\Gamma(\boldsymbol{x}') \quad (18)$$

and boundary condition (Equation 4)

$$[n_j(\boldsymbol{x}) \oint_{\Gamma} \gamma_k(\boldsymbol{x}')n_{k'}(\boldsymbol{x}')\sigma_{kk'} \times \{\sigma_{ij}[\mathbf{G}(\boldsymbol{x}, \boldsymbol{x}')]\}\mathrm{d}\Gamma(\boldsymbol{x}') - n_j(\boldsymbol{x})\sigma_{ij}(\boldsymbol{I})] = \sigma_{ij}^{0}(\boldsymbol{x})n_j(\boldsymbol{x}) + n_j(\boldsymbol{x}) \sum_{\alpha,\beta} C_{\alpha\beta}^{ij} x_1^{\alpha} x_2^{\beta} \quad (\boldsymbol{x} \subset \Gamma) \quad (19)$$

In the above relations we have used the notation

$$\boldsymbol{I} \equiv I_{\mathrm{m}} = -\oint_{G} \sigma_{ij,j}^{*\mathrm{II}}(\boldsymbol{x}')G_{\mathrm{im}}(\boldsymbol{x}, \boldsymbol{x}')\mathrm{d}\Omega(\boldsymbol{x}') \quad (20)$$

Also, the part of the Ω-integral in Equation 4 arising from $\sigma_{ij}^{*\mathrm{I}}$ is easily seen to lead to the last-term on the right hand side of Equation 19 and it differs from the integral in Equation 13 only to the extent that σ_{ij} replaces the $\bar{\sigma}_{ij}$-operator. Thus $C_{\alpha\beta}^{ij}$ in Equation 19 are given by a modified from of Equation 14, i.e.

$$C_{\alpha\beta}^{ij} = B_{pq}^{kk'} D_{kk'ij}^{pq\alpha\beta} \quad (21)$$

where $D_{kk'ij}^{pq\alpha\beta}$ are obtained from $\bar{D}_{kk'ij}^{pq\alpha\beta}$ on replacing $(\bar{\lambda}, \bar{\mu})$ in the latter by (λ, μ).

The solution of Equations 18 and 19 for $\sigma_{ij}^{*\mathrm{II}}(\boldsymbol{x})$ and $\gamma_k(\boldsymbol{x})$ seems to be far more complicated than that of the anti-plane case [6]. However, when Poisson's ratios of the two materials are equal a procedure similar to that adopted in [6] may be followed.

3.2. Special case of equal Poisson's ratios ($\nu = \nu_1$)

When $\nu = \nu_1$, assume that the eigenstrains $e_{ij}^{*\mathrm{II}}$ and eigenstresses $\sigma_{ij}^{*\mathrm{II}}$ can be derived from a "generating" displacement $\boldsymbol{u}^{*\mathrm{II}}$ of the form

$$u^{*\mathrm{II}}(x) \equiv u_m^{*\mathrm{II}}(x) = -\oint_\Gamma B_k(x')n_{k'}(x')\sigma_{kk'}\{G^m(x, x')\}\mathrm{d}\Gamma(x') \tag{22}$$

This is justified because the singular part will arise only from a distribution of sources (dislocations) on the debonded arc Γ. Substituting Equation 22 into Equation 20 leads to the result

$$I_m = \int_\Gamma B_k(x'')n_{k'}(x'')\sigma_{kk'}(\xi)\mathrm{d}\Gamma(x'') \tag{23}$$

where

$$\begin{aligned}\xi &= \int_\Omega \sigma_{ij,j}[\mathrm{G}(x', x'')]G_{im}(x, x')\mathrm{d}\Omega(x') \\ &= \int_\Gamma [-\eta_i\delta(x' - x'')]G_{im}(x, x')\mathrm{d}\Omega(x') \\ &= -\eta_i G_{im}(x, x'') = -G^m(x, x'')\end{aligned} \tag{24}$$

Here we have used the fact that

$$\sigma_{ij,j}(G^m) = -\delta_{im}\delta(x - x') \tag{25}$$

and have defined

$$\eta_i = (\delta_{i1}, \delta_{i2}), \qquad i = 1, 2 \tag{26}$$

In Equation 25 G^m denotes the vector $G_{km}(x, x')$ at a fixed m, δ_{im} is the Knonecker delta and $\delta(x - x')$ is the Dirac delta function. Employing Equation 24 in Equation 23, we get

$$I_m = -\oint_\Gamma B_k(x')n_{k'}(x')\sigma_{kk'}[G^m(x, x')]\mathrm{d}\Gamma(x') \tag{27}$$

Substituting Equation 23 into Equation 2 the displacement vector u_m may be written in a simple form

$$\begin{aligned}u_m(x) = &-\oint_\Gamma [B_i(x') + \gamma_i(x')]n_j(x')\sigma_{ij}[G^m(x, x')]\mathrm{d}\Gamma(x') \\ &+ \int_\Omega \sigma_{ij}^{*\mathrm{I}}(x')G_{im,j}(x, x')\mathrm{d}\Omega(x')\end{aligned} \tag{28}$$

Substituting Equations 22 and 27 into the equivalence relation (Equation 18) gives

$$\begin{aligned}&\oint_\Gamma B_k(x')n_{k'}(x')\sigma_{ij}\{\sigma_{kk'}[\mathrm{G}(x, x')]\}\mathrm{d}\Gamma(x') \\ &= \oint_\Gamma [B_k(x') + \gamma_{k'}(x')]\, n_{k'}(x') \\ &\times \bar{\sigma}_{ij}\{\sigma_{kk'}[\mathrm{G}(x, x')]\}\, \mathrm{d}\Gamma(x') \qquad (x \subset \Omega)\end{aligned} \tag{29}$$

When $\nu = \nu_1$ (i.e. $\lambda/\mu = \lambda_1/\mu_1$), we have

$$\bar{\lambda}/\lambda = \bar{\mu}/\mu \tag{30}$$

In this special case the operators σ_{ij} and $\bar{\sigma}_{ij}$ are linear multiples of each other (see Equation 9), and Equation 29 has the simple solution

$$B_k(x) = \frac{\lambda - \lambda_1}{\lambda_1}\gamma_k(x) \tag{31}$$

The second relation between $B_k(x)$ and $\gamma_k(x)$ is provided by the boundary conditions (Equation 19). From Equations 19 and 27 it follows that

$$\begin{aligned}&n_j(x)\oint_\Gamma \{B_k(x') + \gamma_k(x')\}n_{k'}(x') \\ &\times \sigma_{ij}\{\sigma_{kk'}[\mathrm{G}(x, x')]\}\, \mathrm{d}\Gamma(x') = \sigma_{ij}^0(x)n_j(x) \\ &+ n_j(x)\sum_{\alpha,\beta} C_{\alpha\beta}^{ij}x_1^\alpha x_2^\beta \qquad (x \subset \Gamma)\end{aligned} \tag{32}$$

Note that on the debonded arc Γ

$$x_1 = a(1 - x_2^2/b^2)^{\frac{1}{2}}, \qquad -l \leqslant x_2 \leqslant l \tag{33}$$

where $2l$ measures the span of the debonded region and a and b are the semi-major and semi-minor axes of the elliptic inhomogeneity.

The unit outward normal n_i on Γ is defined by

$$(n_1, n_2) = (x_1/a^2, x_2/b^2)/T(x) \tag{34}$$

where

$$T(x) = (x_1^2/a^4 + x_2^2/b^4)^{\frac{1}{2}} \tag{35}$$

Also note that, on Γ,

$$\mathrm{d}\Gamma(x') = T(x')(a^2/x_1')\mathrm{d}x_2' \tag{36}$$

In view of the expected singular behaviour of the solution, we assume that

$$\begin{aligned}[B_k(x') + \gamma_k(x')] &= \frac{1}{2\pi}(l^2 - x_2'^2)^{\frac{1}{2}} \\ &\times (\gamma_0^k + \gamma_1^k x_2' + \gamma_2^k x_2'^2 + \ldots)\end{aligned} \tag{37}$$

Thus Equation 32 reduces to a singular integral equation with Cauchy-type kernel

$$\begin{aligned}&\frac{1}{2\pi}\int_{-l}^{l} \frac{(l^2 - x_2'^2)^{\frac{1}{2}}F_i(x_2, x_2')}{(x_2' - x_2)^2}\mathrm{d}x_2' \\ &= p_i(x_2)(-l < x_2 < l); \qquad i = 1, 2\end{aligned} \tag{38}$$

where

$$F_i(x_2, x_2') = (\gamma_0^k + \gamma_1^k x_2' + \gamma_2^k x_2'^2 + \ldots) \times T(\boldsymbol{x}) n_j(\boldsymbol{x}) n_{k'}(\boldsymbol{x}') \frac{\mathrm{d}\Gamma(x')}{\mathrm{d}x_2'} \times \sigma_{ij}\{\sigma_{kk'}[\mathbf{G}(\boldsymbol{x}, \boldsymbol{x}')]\} (x_2' - x_2)^2 \quad (39)$$

It can be shown that

$$F_i(x_2, x_2') = \sum_{m,n} \alpha_{mn}^{(i)} x_2^m x_2'^n \quad (40)$$

where $\alpha_{mn}^{(i)}$ are defined in terms of γ_n^k in the Appendix. In Equation 38,

$$p_i(x_2) = (x_1/a^2)\sigma_{i1}^0 + (x_2/b^2)\sigma_{i2}^0 + (x_1/a^2)(C_{00}^{i1} + C_{10}^{i1} x_1 + C_{01}^{i1} x_2 + \ldots) + (x_2/b^2)(C_{00}^{i2} + C_{10}^{i2} x_1 + C_{01}^{i2} x_2 + \ldots) \quad (41)$$

where x_1 is defined by Equation 33.

Employing Equations 40 and 41 in Equation 38, the integration can be performed term by term, and a comparison of both sides made after expanding them in power series of x_2 to give the required relations for γ_n^k. Up to the first order terms, these are

$$\left[\gamma_0^k(-\tfrac{1}{2} r_{00}^{ik} + \tfrac{1}{4} l^2 r_{02}^{ik}) + \gamma_1^k\left(\frac{l^2}{4} r_{01}^{ik}\right) + \gamma_2^k\left(\frac{l^2}{4} r_{00}^{ik}\right)\right] = \frac{1}{a}(A_{00}^{i1} + C_{00}^{i1}) + (A_{10}^{i1} + C_{10}^{i1}); \quad (42)$$

$$\left[\gamma_0^k\left(-r_{01}^{ik} - \tfrac{1}{2} r_{10}^{ik} + \frac{l^2}{4} r_{12}^{ik}\right) + \gamma_1^k\left(-r_{00}^{ik} + \frac{l^2}{4} r_{11}^{ik}\right) + \gamma_2^k\left(\frac{l^2}{4} r_{10}^{ik}\right)\right] = \frac{1}{b^2}(A_{00}^{i2} + C_{00}^{i2}) + a\left(\frac{A_{01}^{i1}}{a^2} + \frac{A_{10}^{i2}}{b^2} + \frac{C_{01}^{i1}}{a^2} + \frac{C_{10}^{i2}}{b^2}\right); \quad (43)$$

$$\left[\gamma_0^k\left(-\tfrac{3}{2} r_{02}^{ik} - r_{11}^{ik} - \tfrac{1}{2} r_{20}^{ik} + \frac{l^2}{4} r_{22}^{ik}\right) + \gamma_1^k\left(-\tfrac{3}{2} r_{01}^{ik} - r_{10}^{ik} + \frac{l^2}{4} r_{21}^{ik}\right) + \gamma_2^k\left(-\tfrac{3}{2} r_{00}^{ik} + \frac{l^2}{4} r_{20}^{ik}\right)\right] = -\frac{1}{2ab^2}\left(A_{00}^{i1} + C_{00}^{i1}\right) - \frac{1}{b^2}\left(A_{10}^{i1} + C_{10}^{i1}\right) + \frac{1}{b^2}\left(A_{01}^{i2} + C_{01}^{i2}\right) \quad (44)$$

Here, $i, k = 1, 2$, and summation over repeated indices is assumed. r_{mn}^{ik} are defined in the Appendix, while $A_{\alpha\beta}^{ij}$ are given by Equation 17. This completes the determination of the singular part of the eigenstrains $e_{ij}^{*\mathrm{II}}$.

4. Stress-intensity factors and relative displacements of the debond faces

The singular behaviour of the stresses at the tips of the debonded arc can be studied by applying the stress-operator to Equation 28 and retaining only the singular parts given by the integral over Γ:

$$\tau_i \equiv \sigma_{ij} n_j = -n_j \oint_\Gamma [B_k(\boldsymbol{x}') + \gamma_k(\boldsymbol{x}')] n_{k'}(\boldsymbol{x}') \times \sigma_{ij}\{\sigma_{kk'}[\mathbf{G}(\boldsymbol{x}, \boldsymbol{x}')]\} \mathrm{d}\Gamma(\boldsymbol{x}')$$

$$= -\frac{1}{2\pi T(\boldsymbol{x})} \oint_\Gamma \frac{(l^2 - x_2'^2)^{\frac{1}{2}} F_i(x_2, x_2')}{(x_2' - x_2)^2} \mathrm{d}x_2'$$

$$= -\frac{1}{2\pi T(\boldsymbol{x})} \oint_\Gamma \frac{(l^2 - x_2'^2)^{\frac{1}{2}}\left(\sum_{m,n} \alpha_{mn}^{(i)} x_2^m x_2'^n\right)}{(x_2' - x_2)^2} \mathrm{d}x_2'$$

$$= \frac{-1}{2\pi T(\boldsymbol{x})}[(\alpha_{00}^{(i)} x_2 + \alpha_{01}^{(i)} x_2^2 + \alpha_{02}^{(i)} x_2^3 + \ldots) + (\alpha_{10}^{(i)} x_2^2 + \alpha_{11}^{(i)} x_2^3 + \alpha_{12}^{(i)} x_2^4 + \ldots) + (\alpha_{20}^{(i)} x_2^3 + \alpha_{21}^{(i)} x_2^4 + \alpha_{22}^{(i)} x_2^5 + \ldots) + \ldots] \frac{1}{(x_2^2 - l^2)^{\frac{1}{2}}}; \quad (|x_2| \to l + 0) \quad (45)$$

Resolving τ_i in the normal (n_i) and tangential (t_i) directions and letting $x_2 \to \pm l$ we obtain the stress-intensity factors at the debond tips as follows. First we write the normal and tangential stresses as

$$\sigma_{nn} = \tau_i n_i = K_{\mathrm{I}}^{\pm}/(2\pi d_0)^{\frac{1}{2}} \quad (46)$$

$$\sigma_{nt} = \tau_i t_i = K_{\mathrm{II}}^{\pm}/(2\pi d_0)^{\frac{1}{2}} \quad (47)$$

where $\pm$ refer to $x_2 = \pm l$ and d_0 is the distance along the tangent from either of the tips. The stress-intensity factors $K_{\mathrm{I}}^{\pm}$ and $K_{\mathrm{II}}^{\pm}$ are given by

$$K_{\mathrm{I}}^{\pm} = K_1^{\pm} n_1^{\pm} + K_2^{\pm} n_2^{\pm} \quad (48)$$

$$K_{\mathrm{II}}^{\pm} = K_1^{\pm} t_1^{\pm} + K_2^{\pm} t_2^{\pm} \quad (49)$$

where

$$n^{\pm} \equiv (n_1^{\pm}, n_2^{\pm})$$
$$= \left[\left(1 - \frac{l^2}{b^2}\right)^{\frac{1}{2}}, \pm a\frac{l}{b^2}\right] \Big/ \left(1 - \frac{l^2}{b^2} + a^2\frac{l^2}{b^4}\right)^{\frac{1}{2}}$$
$$t^{\pm} \equiv (t_1^{\pm}, t_2^{\pm})$$
$$= \left[-\frac{al}{b^2}, \pm\left(1 - \frac{l^2}{b^2}\right)^{\frac{1}{2}}\right] \Big/ \left(1 - \frac{l^2}{b^2} + \frac{a^2 l^2}{b^4}\right)^{\frac{1}{2}} \quad (50)$$

and

$$K_i^{\pm} = \pm\frac{(\pi l)^{\frac{1}{2}} a(1 + a^2 l^2/b^4)^{\frac{1}{4}}}{2\left(1 - \frac{l^2}{b^2} + \frac{a^2 l^2}{b^4}\right)^{\frac{1}{2}}} \times \{\alpha_{00}^{(i)} \pm [\alpha_{01}^{(i)} + \alpha_{10}^{(i)}]\, l + [\alpha_{02}^{(i)} + \alpha_{11}^{(i)} + \alpha_{20}^{(i)}]\, l^2 + \ldots\} \quad (i = 1, 2) \quad (51)$$

where $\alpha_{mn}^{(i)}$ are defined in the Appendix. The non-dimensional stress-intensity factors are defined by

$$\bar{K}_{\rm I,II}^{\pm} = K_{\rm I,II}^{\pm}/(\mu l^{\frac{1}{2}}) \quad (52)$$

The "effective" relative displacement of the debonded faces is given by

$$\Delta\bar{u}_k = B_k(x) + \gamma_k(x)$$
$$= \frac{1}{2\pi}(l^2 - x_2^2)^{\frac{1}{2}}(\gamma_0^k + \gamma_1^k x_2 + \gamma_2^k x_2^2 + \ldots) \quad (53)$$

$\Delta\bar{u}_k$ can be resolved in the normal and tangential directions to yield

$$(\Delta\bar{u})_{n\pm} = \Delta\bar{u}_k n_{k\pm} \quad (54)$$

$$(\Delta\bar{u})_{t\pm} = \Delta\bar{u}_k t_{k\pm} \quad (55)$$

5. Numerical examples

We illustrate the procedure by means of a specific example. The applied strain (Equation 10) we take in the form

$$e_{ij}^0 = (a_{00}^{ij} + a_{10}^{ij} x_1 + a_{01}^{ij} x_2) \quad (56)$$

where we have retained terms only up to first order in x. We assume correspondingly the non-singular part of eigenstrains:

$$e_{ij}^{*\rm I} = (b_{00}^{ij} + b_{10}^{ij} x_1 + b_{01}^{ij} x_2) \quad (57)$$

The Relations 14 defining $\bar{C}_{mn}^{ij}$ for this case are explicitly given in the Appendix. $B_{\alpha\beta}^{ij}$ can be solved from Equation 16, and b_{mn}^{ij} from Equation 15. Thus the eigenstrains $e_{ij}^{*\rm I}$ in Equation 57 are determined.

Next C_{mn}^{ij} are obtained by merely replacing ($\bar{\lambda}$, $\bar{\mu}$) by (λ, μ) in the expressions for $\bar{D}_{kk'ij}^{n}$ given the the Appendix. This allows the determination of γ_n^k from Equations 42 to 44. From Equations 31 and 37 we finally obtain the functions $\gamma_k(x)$ and $B_k(x)$

$$\gamma_k(x) = \frac{1}{2\pi}\frac{\lambda_1}{\lambda}(l^2 - x_2^2)^{\frac{1}{2}} \times (\gamma_0^k + \gamma_1^k x_2 + \gamma_2^k x_2^2 + \ldots) \quad (58)$$

$$B_k(x) = \frac{1}{2\pi}\frac{\lambda - \lambda_1}{\lambda}(l^2 - x_2^2)^{\frac{1}{2}} \times (\gamma_0^k + \gamma_1^k x_2 + \gamma_2^k x_2^2 + \ldots) \quad (59)$$

Using $\alpha_{mn}^{(i)}$ obtained in the Appendix, the stress-intensity factors can be found from Equations 51 and 52. The relative displacements across the debonded arc Γ are given by Equations 53 to 55.

For the numerical calculations we have considered the following two loading cases.

Case 1

$$e_{11}^0 = 1, \quad e_{22}^0 = e_{12}^0 = 0 \quad (60)$$

For this case, it follows from Equation 56

$$a_{00}^{11} = 1$$
$$a_{10}^{11} = a_{01}^{11} = 0$$
$$a_{mn}^{12} = a_{mn}^{22} = 0, \quad \text{for all } m \text{ and } n. \quad (61)$$

The solution is symmetric about $x_2 = 0$. Hence the stress-intensity factors and relative displacements are shown only for the region $x_2 > 0$.

Figs. 2 and 3 show the variation for $\bar{K}_{\rm I}$ and $\bar{K}_{\rm II}$ with respect to the debond size (l/b) for several values of the ratio (b/a) of the elliptic inhomogeneity and for two values of the parameter $\lambda_1/\lambda = 20$ and 50. Here, and in the sequel, Poisson's ratio has been chosen as $\nu = \nu_1 = 0.3$. The debond size affects the stress-intensity factors significantly, especially for large values of b/a.

Figs. 4 and 5 show the normal and tangential relative displacements across Γ for $b/a = 0.4$ and 0.8, respectively. In both cases, we have assumed $\lambda_1/\lambda = 20$ and $\nu = \nu_1 = 0.3$.

Case 2

$$e_{11}^0 = 1 + x_1 + x_2, \quad e_{12}^0 = e_{22}^0 = 0 \quad (62)$$

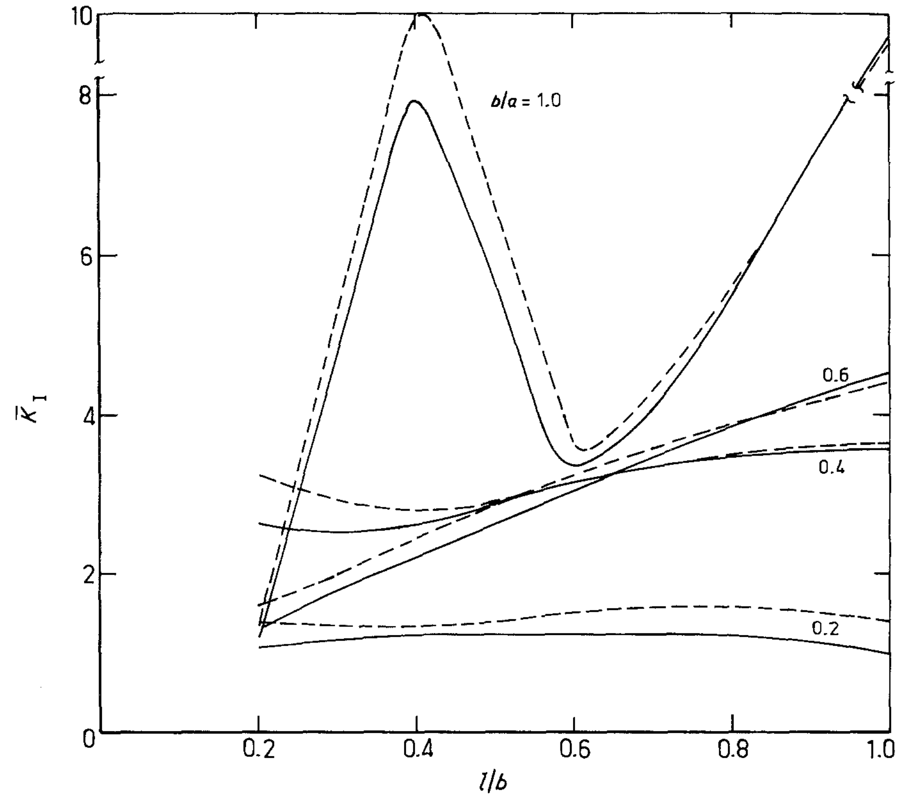

Figure 2 Stress-intensity factor $\bar{K}_I$ corresponding to the applied strain (Equation 60). $\lambda_1/\lambda = 20$ (solid lines) and $\lambda_1/\lambda = 50$ (broken lines). In all cases, $\nu = \nu_1 = 0.3$.

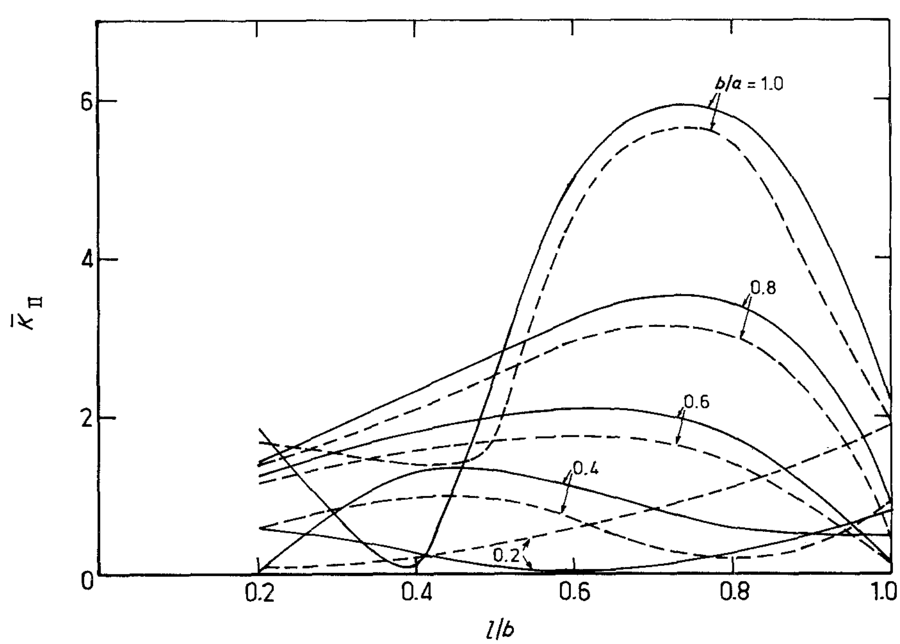

Figure 3 Stress-intensity factor $\bar{K}_{II}$ arising from Equation 60. Key as in Fig. 2.

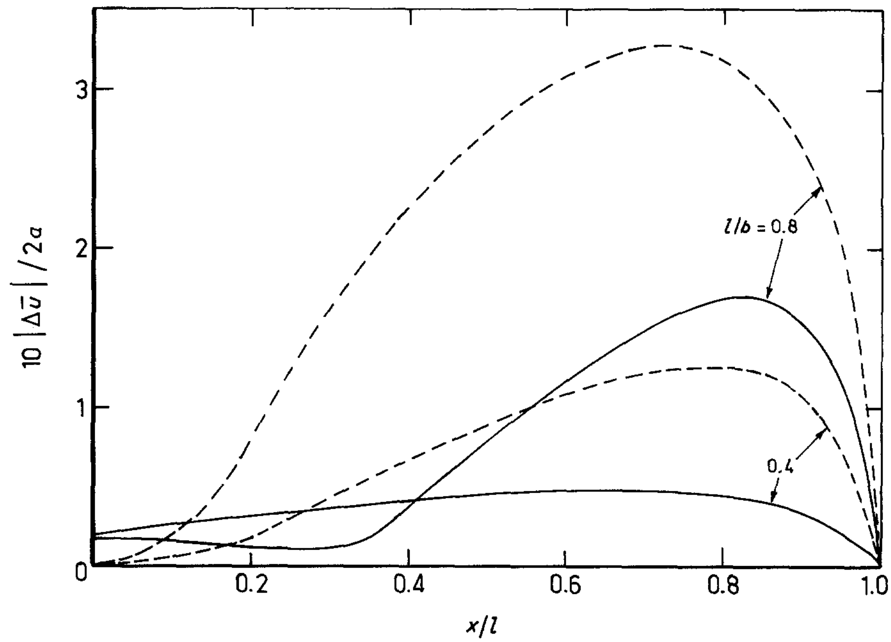

Figure 4 Relative displacements of the debonded faces due to Equation 60. $\lambda_1/\lambda = 20$, $b/a = 0.4$ and $\nu = \nu_1 = 0.3$. Normal displacements are shown by solid lines and tangential displacements by broken lines.

In this case,

$$a_{00}^{11} = 1, \quad a_{10}^{11} = 1, \quad a_{01}^{11} = 1$$

$$a_{mn}^{12} = a_{mn}^{22} = 0, \quad \text{for all } m \text{ and } n \quad (63)$$

It is obvious that the results will not be symmetric about $x_2 = 0$. Figs. 6 and 7 show the stress-intensity factors on either side of $x_2 = 0$ for $\lambda_1/\lambda = 20$, and $\nu = \nu_1 = 0.3$. Again, it is noticed that for large values of b/a, the fluctuation with respect to the debond size (l/b) is prominent.

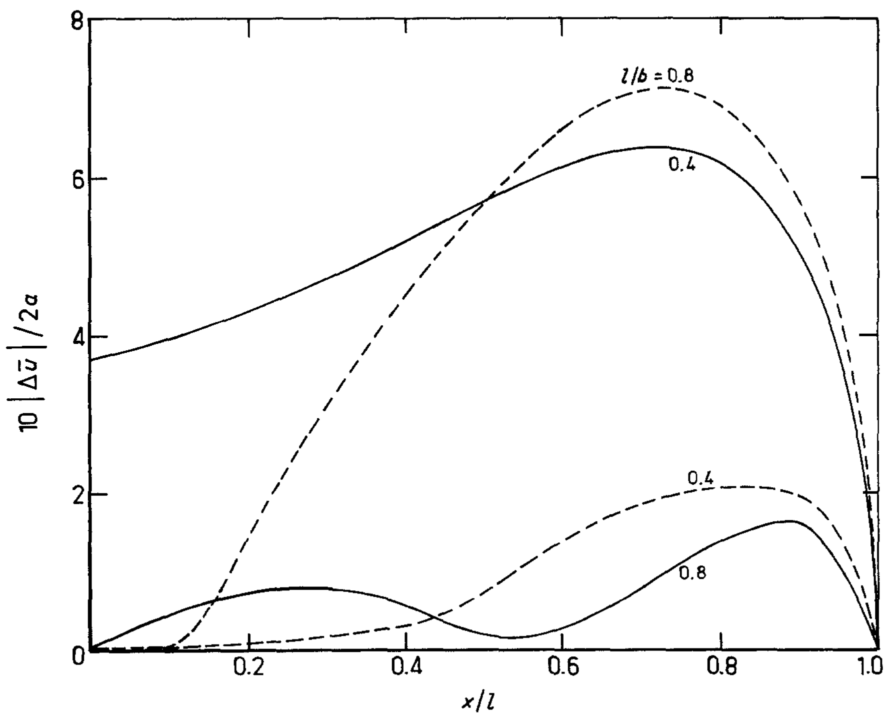

Figure 5 Relative displacements of the debonded faces due to Equation 60. $\lambda_1/\lambda = 20$, $b/a = 0.8$ and $\nu = \nu_1 = 0.3$.

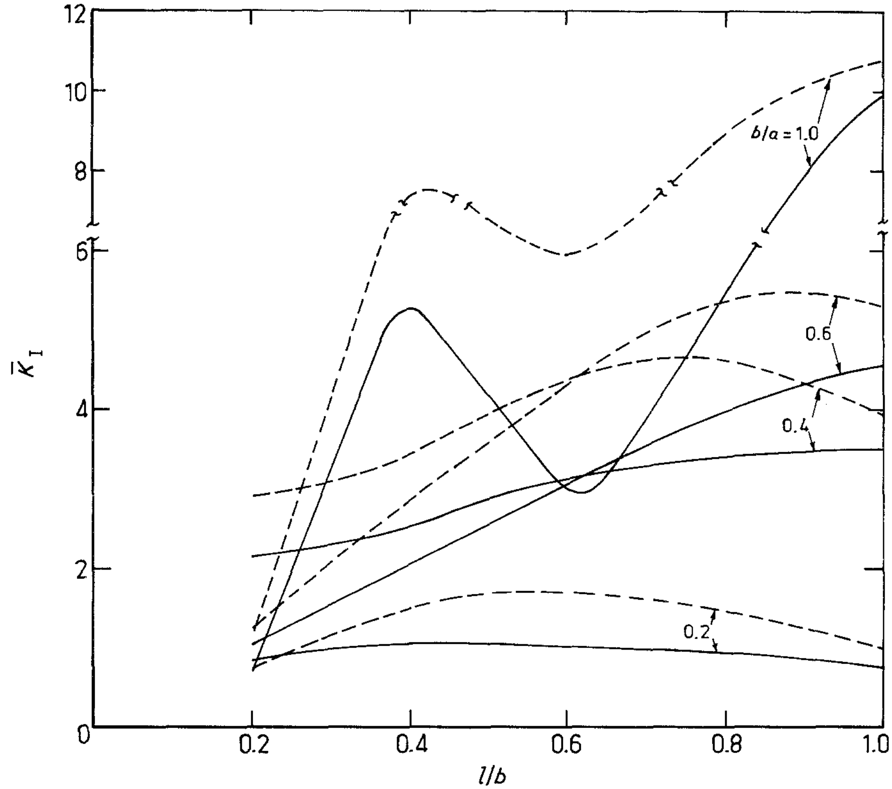

Figure 6 The stress-intensity factor $\bar{K}_I$ for the applied strain (Equation 62). $\lambda_1/\lambda = 20$, and $\nu = \nu_1 = 0.3$. (Solid lines represent $x_2 > 0$ side of debond and broken lines $x_2 < 0$ side of the debond.)

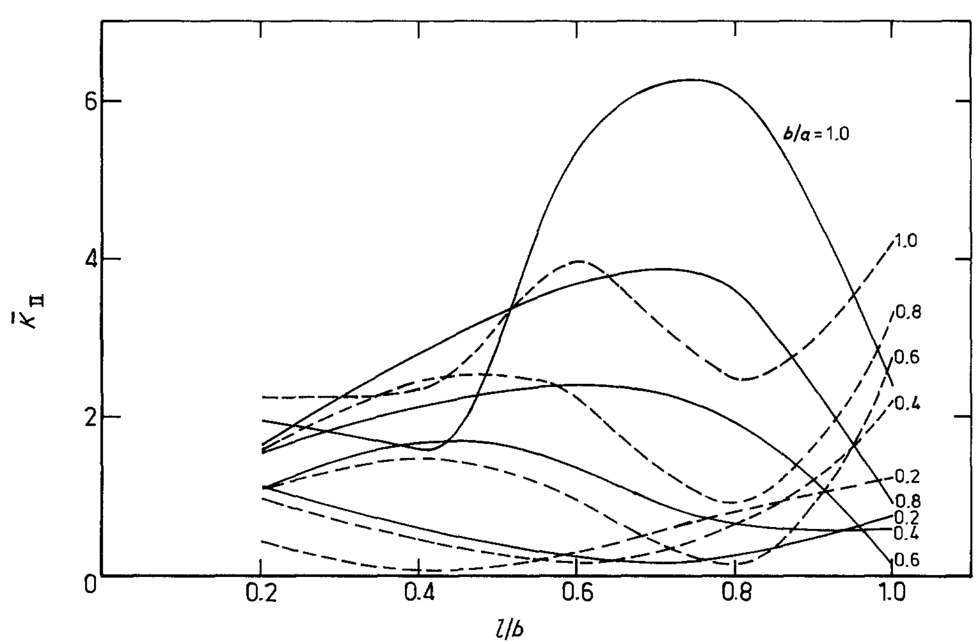

Figure 7 The stress-intensity factor $\bar{K}_{II}$ for the applied strain (Equation 62). $\lambda_1/\lambda = 20$, and $\nu = \nu_1 = 0.3$. (Solid lines represent $x_2 > 0$ side of the debond and broken lines $x_2 < 0$ side of the debond.)

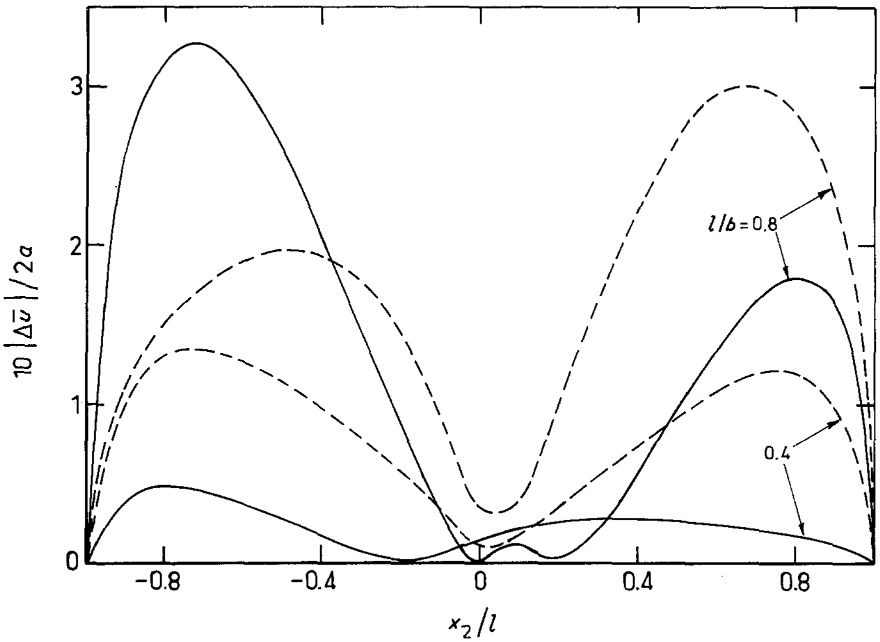

Figure 8 Relative displacements across the debonded faces corresponding to the strain (Equation 62). $\lambda_1/\lambda = 20$, $b/a = 0.4$, $\nu = \nu_1 = 0.3$. Normal displacements are shown by solid lines and tangential displacements by broken lines.

Figs. 8 and 9 show the relative displacements across the debond Γ for $-l \leqslant x_2 \leqslant l$ for the two cases $b/a = 0.4$ and 0.8. In both cases, we have chosen $\lambda_1/\lambda = 20$ and $\gamma = \gamma_1 = 0.3$. Apparently peak values occur near the tips for large values of the ratio l/b. This may be due to the restriction on the Poisson ratio.

In conclusion it may be noted that the stress-

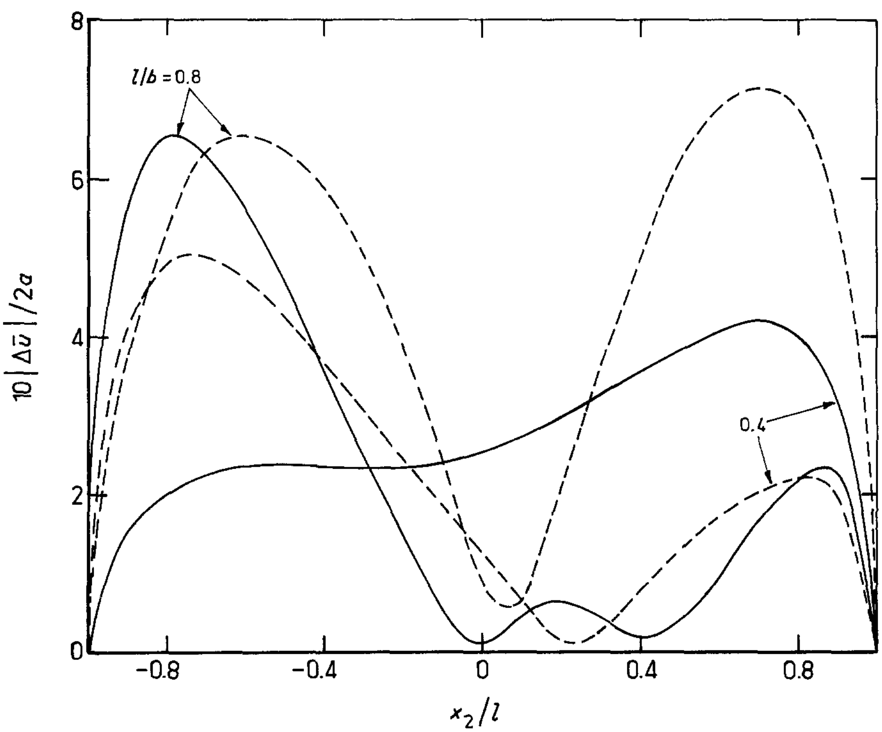

Figure 9 Relative displacements across the debonded faces corresponding to the strain (Equation 62). $\lambda_1/\lambda = 20$, $b/a = 0.8$, $\nu = \nu_1 = 0.3$. Normal displacements are shown by solid lines and tangential displacements by broken lines.

intensity factors at the debond tips and the relative displacement across the debond are greatly influenced by any inhomogeneity in the external stress field and by the extent of the debond size. The present analysis was restricted to the matrix and inclusion having the same Poisson's ratio. The influence of relaxing this restriction is under investigation and will be reported in a separate communication.

Appendix

The integral Equation 13 may be formally rewritten as

$$I_{ij} = \bar{\sigma}_{ij}(\boldsymbol{U}) = \int_{\Omega} \sigma^{*I}_{kl}(\boldsymbol{x}')\bar{\sigma}_{ij}\{\boldsymbol{G}^k(\boldsymbol{x}, \boldsymbol{x}')\}_{,l}\mathrm{d}\Omega(\boldsymbol{x}') \quad \text{(A1)}$$

where

$$\boldsymbol{U} \equiv U_i = \int_{\Omega} \sigma^{*I}_{kl}(\boldsymbol{x}')G_{ik,l}(\boldsymbol{x}, \boldsymbol{x}')\mathrm{d}\Omega(\boldsymbol{x}')$$

and

$$\sigma^{*I}_{kl}(\boldsymbol{x}) = \sum_{\alpha,\beta} B^{kl}_{\alpha\beta}x_1^{\alpha}x_2^{\beta} = B^{kl}_{00} + B^{kl}_{10}x_1 + B^{kl}_{01}x_2 \quad \text{(A2)}$$

In Equation A2 we have retained only terms upto the first order in $\boldsymbol{x}$ as required by the examples considered in Section 5. It should be noted that I_{ij} (Equation A1) will be a polynomial of the same order as σ^{*I}_{ij} (Equation A2):

$$I_{ij} = \bar{C}^{ij}_{00} + \bar{C}^{ij}_{10}x_1 + \bar{C}^{ij}_{01}x_2 \quad \text{(A3)}$$

where

$$\begin{aligned}\bar{C}^{ij}_{00} &= B^{kl}_{00}\bar{D}^{1}_{klij}\\ \bar{C}^{ij}_{10} &= B^{kl}_{10}\bar{D}^{2}_{klij} + B^{kl}_{01}\bar{D}^{3}_{klij}\\ \bar{C}^{ij}_{01} &= B^{kl}_{10}\bar{D}^{4}_{klij} + B^{kl}_{01}\bar{D}^{5}_{klij}\end{aligned} \quad \text{(A4)}$$

The coefficients $\bar{D}^m_{klij}$, which are written here in a more simplified form than in the text (Equation 14) because of the order of $\boldsymbol{x}$ appearing Equation A2, are obtained by following a procedure similar to that adopted in [6]. In fact, it can be shown that

$$\bar{D}^m_{klij} = \bar{\lambda}\delta_{ij}(T^m_{kl11} + T^m_{kl22}) + \bar{\mu}(T^m_{klij} + T^m_{klji}) \quad \text{(A5)}$$

where

$$\begin{aligned}
T^1_{klij} = A\Big[&-(3-4\nu)(I_1d_1 + I_2d_2)\\
&+ I_1(d_3+d_5) + I_2(d_4+d_6)\\
&- \frac{2\delta_{j1}}{a^2}Q^{kli}_1 - \frac{2\delta_{j2}}{b^2}Q^{kli}_2\Big]\\
T^2_{klij} = A\Big[&-(3-4\nu)(3I_1 - 2I_3)d_1\\
&-(3-4\nu)(I_2 - 2I_4)d_2\\
&+(3I_1 - 2I_3)(d_3+d_5)\\
&+(I_2 - 2I_4)(d_4+d_6)\\
&- \frac{4\delta_{j1}}{a^2}Q^{kli}_3 - \frac{2\delta_{j2}}{a^2b^2}Q^{kli}_4\Big]\\
T^3_{klij} = A\Big[&-(3-4\nu)(I_1 - 2I_4)d_7\\
&-(3-4\nu)(I_2 - 2I_4)(b^2/a^2)d_8\\
&+(I_1 - 2I_4)(d_9 + d_{11})\\
&+(I_2 - 2I_4)(d_{10} + d_{12})\\
&- \frac{4\delta_{j1}}{a^2}Q^{kli}_{10} - \frac{2\delta_{j2}}{a^2b^2}Q^{kli}_{11}\Big]\\
T^4_{klij} = A\Big[&-(3-4\nu)(a^2/b^2)(I_1 - 2I_4)d_7\\
&-(3-4\nu)(I_2 - 2I_4)d_8\\
&+(I_1 - 2I_4)(d_9 + d_{11})\\
&+(I_2 - 2I_4)(d_{10} + d_{12})\\
&- \frac{2\delta_{j1}}{a^2b^2}Q^{kli}_4 - \frac{4\delta_{j2}}{b^2}Q^{kli}_5\Big]\\
T^5_{klij} = A\Big[&-(3-4\nu)(I_1 - 2I_4)d_1\\
&-(3-4\nu)(3I_2 - 2I_5)d_2\\
&+(I_1 - 2I_4)(d_3 + d_5)\\
&+(3I_2 - 2I_5)(d_4 + d_6)\\
&- \frac{2\delta_{j1}}{a^2b^2}Q^{kli}_{11} - \frac{4\delta_{j2}}{b^2}Q^{kli}_{12}\Big]
\end{aligned} \quad \text{(A6)}$$

and

$$A = -1/[4\mu(1-\nu)]$$
$$(d_1, d_7) = (\delta_{j1}, \delta_{j2})\delta_{ki}\delta_{l1}$$
$$(d_2, d_8) = (\delta_{j2}, \delta_{j1})\delta_{ki}\delta_{l2}$$
$$(d_3, d_9) = (\delta_{j1}, \delta_{j2})\delta_{kl}\delta_{i1} \quad \text{(A7)}$$
$$(d_4, d_{10}) = (\delta_{j2}, \delta_{j1})\delta_{kl}\delta_{i2}$$
$$(d_5, d_{11}) = (\delta_{j1}, \delta_{j2})\delta_{li}\delta_{k1}$$
$$(d_6, d_{12}) = (\delta_{j2}, \delta_{j1})\delta_{li}\delta_{k2}$$

The various integrals $I_1, \ldots, I_5$ and $Q_1^{kli}, \ldots, Q_{13}^{kli}$ appearing in Equations A6 are given by

$$I_1 = (b/a)/(1 + b/a)$$
$$I_2 = 1/(1 + b/a)$$
$$I_3 = (b^2/a^2)(1 + a/2b)/(1 + b/a)^2$$
$$I_4 = (b/a)/[2(1 + b/a)^2]$$
$$I_5 = (1 + b/2a)/(1 + b/a)^2 \quad \text{(A8)}$$
$$Q_1^{kli} = \frac{1}{2\pi}\int_0^{2\pi} l_1 \frac{l_k l_l l_i}{\boldsymbol{l}} \mathrm{d}\phi$$
$$Q_2^{kli} = \frac{1}{2\pi}\int_0^{2\pi} l_2 \frac{l_k l_l l_i}{\boldsymbol{l}} \mathrm{d}\phi$$
$$Q_3^{kli} = \tfrac{3}{2}Q_1^{kli} - \frac{1}{a^2}Q_7^{kli}$$
$$Q_4^{kli} = a^2 Q_2^{kli} - 2Q_8^{kli}$$
$$Q_5^{kli} = \tfrac{1}{2}Q_1^{kli} - \frac{1}{b^2}Q_9^{kli}$$
$$Q_6^{kli} = -\tfrac{1}{2}Q_1^{kli}$$
$$Q_7^{kli} = \frac{1}{2\pi}\int_0^{2\pi} l_1^3 \frac{l_k l_l l_i}{\boldsymbol{l}^2} \mathrm{d}\phi$$
$$Q_8^{kli} = \frac{1}{2\pi}\int_0^{2\pi} l_1^2 l_2 \frac{l_k l_l l_i}{\boldsymbol{l}^2} \mathrm{d}\phi$$
$$Q_9^{kli} = \frac{1}{2\pi}\int_0^{2\pi} l_1 l_2^2 \frac{l_k l_l l_i}{\boldsymbol{l}^2} \mathrm{d}\phi \quad \text{(A9)}$$
$$Q_{10}^{kli} = \tfrac{1}{2}Q_2^{kli} - \frac{1}{a^2}Q_8^{kli}$$
$$Q_{11}^{kli} = b^2 Q_1^{kli} - 2Q_9^{kli}$$
$$Q_{12}^{kli} = \tfrac{3}{2}Q_2^{kli} - \frac{1}{b^2}Q_{13}^{kli}$$
$$Q_{13}^{kli} = \frac{1}{2\pi}\int_0^{2\pi} l_2^3 \frac{l_k l_l l_i}{\boldsymbol{l}^2} \mathrm{d}\phi$$

$l_1 = \cos\phi$, $l_2 = \sin\phi$, $\boldsymbol{l} = (l_1^2/a^2) + (l_2^2/b^2)$.

This completes the evaluation of coefficients $\bar{C}_{\alpha\beta}^{ij}$ appearing in Equation 13. To obtain coefficients $C_{\alpha\beta}^{ij}$ appearing in Equation 19, we simply replace $(\bar{\lambda}, \bar{\mu})$ in Equation A5 by (λ, μ).

Next we obtain the expansion of function $F_i(x_2, x_2')$ in powers of x_2 and x_2' (Equation 40). Note that Equation 39 may be rewritten as

$$F_i(x_2, x_2') = (\gamma_0^k + \gamma_1^k x_2' + \gamma_2^k x_2'^2 + \ldots)\phi_{ik}(x_2, x_2') \quad \text{(A10)}$$

where

$$\phi_{ik} = (x_2' - x_2)^2\left[\left(\frac{x_1}{a^2}\sigma_{i1}\{\ \} + \frac{x_2}{b^2}\sigma_{i2}\{\ \}\right) \times \left(\sigma_{k1}\{\ \} + \frac{a^2 x_2'}{b^2 x_1'}\sigma_{k2}\{\ \}\right)\right]\mathbf{G}(\boldsymbol{x}, \boldsymbol{x}') \quad \text{(A11)}$$

Here $\sigma_{ij}\{\ \}$ denotes the usual stress tensor operating on a vector. $\phi_{ik}(x_2, x_2')$ may be expanded in powers of x_2 and x_2' for $\boldsymbol{x}$ and $\boldsymbol{x}'$ on Γ. Accordingly, we write

$$\phi_{ik}(x_2, x_2') = \sum_{\alpha,\beta} r_{\alpha\beta}^{ik} x_2^\alpha x_2'^\beta; \qquad (\boldsymbol{x}, \boldsymbol{x}' \subset \Gamma) \quad \text{(A12)}$$

To determine $r_{\alpha\beta}^{ik}$, we need to evaluate expressions of the following type:

$$\sigma_{ij}\{\ \}\sigma_{k1}\{\ \}\mathbf{G}(\boldsymbol{x}, \boldsymbol{x}') = \sigma_{ij}(v^{kl}), \quad \text{say}$$
$$= \lambda\delta_{ij}(v_{1,1}^{kl} + v_{2,2}^{kl}) + \mu(v_{i,j}^{kl} + v_{j,i}^{kl}) \quad \text{(A14)}$$

where

$$v_{i,j}^{kl} = -\lambda\,\delta_{kl}(G_{1,1j}^i + G_{2,2j}^i) - \nu(G_{k,lj}^i + G_{l,kj}^i) \quad \text{(A15)}$$

the change of sign being due to an interchange in the derivatives with respect to $\boldsymbol{x}$ with those with respect of $\boldsymbol{x}'$. From the definition of Green's function (Equation 6) it follows that

$$G_{m,kl}^i = \frac{1}{8\pi\mu(1-\nu)}\left[\frac{\delta_{lm}\delta_{ki}}{\bar{R}^2} - \frac{2\bar{x}_i\bar{x}_k\delta_{lm}}{\bar{R}^4} + \frac{\delta_{km}\delta_{li}}{\bar{R}^2} - \frac{2\bar{x}_m\bar{x}_k\delta_{li}}{\bar{R}^4} - \frac{(3-4\nu)\delta_{im}\delta_{kl}}{\bar{R}^2} + \frac{2(3-4\nu)\bar{x}_k\bar{x}_l\delta_{im}}{\bar{R}^4} + \frac{8\bar{x}_i\bar{x}_m\bar{x}_l\bar{x}_k}{\bar{R}^6} - \frac{2}{\bar{R}^4}(\delta_{ki}\bar{x}_m\bar{x}_l + \delta_{km}\bar{x}_i\bar{x}_l + \delta_{kl}\bar{x}_i\bar{x}_m)\right] \quad \text{(A16)}$$

where, as before $\bar{x}_i = (x_i' - x_i)$, $\bar{R} = |\boldsymbol{x} - \boldsymbol{x}'|$. Since $x_1 = a(1 - x_2^2/b^2)^{1/2}$ on Γ may be expanded in a series of powers in x_2, let us write

$$\tau_i = \bar{x}_i/\bar{x}_2 = E_{00}^{(i)} + E_{10}^{(i)}x_2 + E_{10}^{(i)}x_2' + \ldots \quad \text{(A17)}$$

where

$$\begin{aligned} E_{00}^{(1)} &= 0, E_{00}^{(2)} = 1, \\ E_{10}^{(1)} &= E_{01}^{(1)} = -a/2b^2, \ldots \\ E_{10}^{(2)} &= E_{01}^{(2)} = 0, \ldots \end{aligned} \quad \text{(A18)}$$

and

$$\begin{aligned} \chi &= \frac{\bar{x}_2^2}{\bar{R}^2} = \frac{1}{1 + \tau_i^2} \\ &= 1 - \frac{a^2}{4b^4}(x_2 + x_2')^2 + \ldots \end{aligned} \quad \text{(A19)}$$

From Equation A17, we also have

$$\tau_i\tau_k = \frac{\bar{x}_i\bar{x}_k}{(\bar{x}_2)^2} = \sum_{m,n} \eta_{mn}^{ik} x_2^m x_2'^n \quad \text{(A20)}$$

where

$$\eta_{\alpha\beta}^{ik} = \sum_{\alpha,\beta} E_{mn}^{(i)} E_{m'n'}^{(i)};$$

$$(m + m' = \alpha, n + n' = \beta) \quad \text{(A21)}$$

Thus we can rewrite Equation A16 as

$$\begin{aligned} (x_2' - x_2)^2 G_{m,kl}^i(\boldsymbol{x}, \boldsymbol{x}') &= \frac{1}{8\pi\mu(1 - v)} \\ \times \{[\delta_{lm}\delta_{ki} + \delta_{km}\delta_{li} &- (3 - 4v)\delta_{im}\delta_{kl}]\chi \\ &- 2\gamma\chi^2 + 8\tau_i\tau_m\tau_k\tau_l\chi^3\} \end{aligned} \quad \text{(A22)}$$

where

$$\begin{aligned} \gamma = \ &\delta_{lm}\tau_i\tau_k + \delta_{li}\tau_m\tau_k - (3 - 4v)\delta_{im}\tau_k\tau_l \\ &+ \delta_{ki}\tau_m\tau_l + \delta_{km}\tau_i\tau_l + \delta_{kl}\tau_i\tau_m \\ &= \sum_{p,q} t_{pq}^{iklm} x_2^p x_2'^q \quad \text{(say)} \end{aligned} \quad \text{(A23)}$$

and

$$\begin{aligned} \tau_i\tau_m\tau_k\tau_l &= \left(\sum_{\alpha,\beta} \eta_{\alpha\beta}^{im} x_2^\alpha x_2'^\beta \sum_{\gamma,\delta} \eta_{\gamma\delta}^{kl} x_2^\gamma x_2'^\delta\right) \\ &\equiv \sum_{p,q} s_{pq}^{imkl} x_2^p x_2'^q \quad \text{(say)} \end{aligned} \quad \text{(A24)}$$

It is easily verified that

$$\begin{aligned} t_{\alpha\beta}^{iklm} = \ &\delta_{lm}\eta_{\alpha\beta}^{ik} + \delta_{li}\eta_{\alpha\beta}^{mk} + \delta_{ki}\eta_{\alpha\beta}^{ml} \\ &+ \delta_{km}\eta_{\alpha\beta}^{il} + \delta_{kl}\eta_{\alpha\beta}^{im} - (3 - 4v)\delta_{im}\eta_{\alpha\beta}^{kl} \end{aligned} \quad \text{(A25)}$$

and

$$s_{pq}^{imkl} = \sum_{\alpha,\beta,\gamma,\delta} \eta_{\alpha\beta}^{im}\eta_{\gamma\delta}^{kl},$$

$$(\alpha + \gamma = p, \quad \beta + \delta = q) \quad \text{(A26)}$$

Substituting Equations A23 and A24 into Equation A22 gives

$$(x_2' - x_2)^2 G_{m,kl}^i = \sum_{\alpha,\beta} F_{\alpha\beta}^{imkl} x_2^\alpha x_2'^\beta \quad \text{(A27)}$$

where

$$\begin{aligned} F_{00}^{imkl} &= A[\Delta_0 - 2t_{00}^{iklm} + 8s_{00}^{imkl}] \\ F_{10}^{imkl} &= A(-2t_{10}^{iklm} + 8s_{10}^{imkl}) \\ F_{01}^{imkl} &= A(-2t_{01}^{iklm} + 8s_{01}^{imkl}) \\ F_{20}^{imkl} &= A\left(-\Delta_0 a^2/4b^4 + \frac{a^2}{b^4}t_{00}^{iklm} - 2\,t_{20}^{iklm} - 6\frac{a^2}{b^4}s_{00}^{imkl} + 8s_{20}^{imkl}\right) \\ F_{02}^{imkl} &= A\left(-\Delta_0 a^2/4b^4 - 2t_{02}^{iklm} + \frac{a^2}{b^4}t_{00}^{iklm} + 8s_{02}^{imkl} - \frac{6a^2}{b^4}s_{00}^{imkl}\right) \\ F_{11}^{imkl} &= A\left(-\Delta_0 a^2/2b^4 + \frac{2a^2}{b^4}t_{00}^{iklm} - 2t_{11}^{iklm} - \frac{12a^2}{b^4}s_{00}^{imkl} + 8s_{11}^{imkl}\right) \end{aligned} \quad \text{(A28)}$$

and

$$\begin{aligned} A &= 1/[8\pi\mu(1 - v)] \\ \Delta_0 &= \delta_{lm}\delta_{ki} + \delta_{km}\delta_{li} - (3 - 4v)\delta_{im}\delta_{kl} \end{aligned} \quad \text{(A29)}$$

Next, using Equation A27 in Equation A15, we have

$$v_{i,j}^{kl} = \frac{1}{(x_2' - x_2)^2}\sum_{\alpha,\beta} f_{\alpha\beta}^{ijkl} x^\alpha x_2'^\beta \quad \text{(A30)}$$

where

$$f_{\alpha\beta}^{ijkl} = -\lambda\delta_{kl}(F_{\alpha\beta}^{i11j} + F_{\alpha\beta}^{i22j}) - \mu(F_{\alpha\beta}^{iklj} + F_{\alpha\beta}^{ilkj}) \quad \text{(A31)}$$

Employing Equation A30 in Equation A12 gives similarly

$$\sigma_{ij}\{\ \}\sigma_{kl}\{\ \}\mathbf{G}(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{(x_2' - x_2)^2}\sum_{\alpha,\beta} \delta_{\alpha\beta}^{ijkl} x_2^\alpha x_2'^\beta \quad \text{(A32)}$$

where

$$\delta_{\alpha\beta}^{ijkl} = \lambda\delta_{ij}(f_{\alpha\beta}^{11kl} + f_{\alpha\beta}^{22kl}) + \mu(f_{\alpha\beta}^{ijkl} + f_{\alpha\beta}^{jikl}) \qquad \text{(A33)}$$

Substituting Equation A32 into Equation A11, we obtain the required power series expansion for $\phi_{ik}(x_2, x_2')$ in the form of Equation A12 where the expansion coefficients are given by

$$\begin{aligned}
r_{00}^{ik} &= (1/a)\delta_{00}^{i1k1} \\
r_{10}^{ik} &= (1/a)\delta_{10}^{i1k1} + (1/b^2)\delta_{00}^{i2k1} \\
r_{01}^{ik} &= (1/a)\delta_{01}^{i1k1} + (1/b^2)\delta_{00}^{i1k2} \qquad \text{(A34)} \\
r_{20}^{ik} &= (1/a)(\delta_{20}^{i1k1} - \frac{1}{2b^2}\delta_{00}^{i1k1}) + (1/b^2)\delta_{10}^{i2k1} \\
r_{11}^{ik} &= (1/a)\delta_{11}^{i1k1} + (1/b^2)\delta_{10}^{i1k2} + (1/b^2)\delta_{01}^{i2k1} \\
&\quad + (a/b^4)\delta_{00}^{i2k2} \\
r_{01}^{ik} &= (1/a)\delta_{02}^{i1k1} + (1/b^2)\delta_{01}^{i1k2}, \qquad \text{etc}
\end{aligned}$$

Finally, it is easy to show from Equations A10 and A12 that

$$F_i(x_2, x_2') = \sum_{m,n} \alpha_{mn}^{(i)} x_2^m x_2'^n \qquad \text{(A35)}$$

where

$$\alpha_{pq}^{(i)} = \sum_{m \leqslant q} \gamma_m^k r_{pm'}^{ik} (m' = q - m) \qquad \text{(A36)}$$

Thus the coefficients r_{mn}^{ij} and $\alpha_{mn}^{(i)}$ are also determined.

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